Cardy-Verlinde formula for an axially symmetric dilaton-axion black hole

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Abstract: It is shown that the Bekenstein-Hawking entropy of an axially symmetric dilaton-axion black hole can be expressed as a Cardy-Verlinde formula. By utilizing the first order quantum correction in the Bekenstein-Hawking entropy we find the modified expressions for the Casimir energy and pure extensive energy. The first order correction to the Cardy-Verlinde formula in the context of axially symmetric dilaton-axion black hole are obtained with the use of modified Casimir and pure extensive energies.

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I. INTRODUCTION

The entropy S_{CFT} of conformal field theory (CFT) in an arbitrary dimension n has been related to its total energy E and Casimir energy E_C by a relation, named as the Cardy-Verlinde formula $S_{CFT} = \frac{2\pi R}{n}\sqrt{E_C(2E - E_C)}$ [1]. The entropy associated with the conformal field theory has been related to the Bekenstein-Hawking entropy for various black hole geometries with asymptotically anti-de Sitter (AdS) boundary [2–10]. Thus, one may naively expect that the entropy of all CFTs that have an AdS-dual description is given as the Cardy-Verlinde formula . However, AdS black holes do not always satisfy the Cardy-Verlinde formula [11]. Recently, much interest has been developed in calculating the quantum corrections to the Bekenstein-Hawking entropy S by using various techniques like radial null geodesics, Hamilton-Jacobi method and loop quantum gravity etc [12–14]. The leading-order correction is proportional to $\ln S$ which comes out to be the same with the use of above techniques. The leading order quantum correction to the classical Cardy-Verlinde formula has been studied by Carlip [15].

The thermodynamics of conformal field theories with gravity duals has been studied actively in literature with the remarkable resemblance of the relevant thermodynamic formulas [1–10]. It has been shown that the Cardy-Verlinde formula holds with a negative cosmological constant or a more general certain potential term for super-gravity scalars [16]. There it has been argued that the Cardy-Verlinde formula also holds for black hole geometry which are asymptotically flat instead of asymptotically AdS space. In the spirit of this Ref. [16], we discuss the entropy of dilaton-axion black hole which is asymptotically flat spacetime in terms of the Cardy-Verlinde formula. Here we consider the stationary axially-symmetric axion-dilaton black hole to study the Cardy-Verlinde formula and its first order correction. This black hole is a string theory inspired black hole in lower spacetime dimensions [17, 18]. The string theory inspired-models consist of two massless scalar fields namely dilaton and axion, in the low energy effective action in four dimension. The thermodynamics of axially-symmetric axion-dilaton black hole is investigated by various authors [19]. We shall demonstrate that the Cardy-Verlinde formula can be related with the Bekenstein-Hawking entropy of the stationary axially-symmetric axion-dilaton black hole. By employing the first order entropy correction to Bekenstein-Hawking entropy, we are able to find the leading order term of the Cardy-Verlinde formula.

The plan of the paper is: In the second section, we shall briefly discuss the thermodynamic quantities associated with the horizon of the stationary axially-symmetric dilaton-axion black hole. In third section, we will study the entropy of the axially-symmetric axion-dilaton black hole which can be represented by the Cardy-Verlinde formula. In the fourth section, we provide the leading order correction to the Cardy-Verlinde formula by using quantum corrected Bekenstein-Hawking entropy in the context of dilaton-axion black hole. Finally we shall conclude our results.

II. AXIALLY SYMMETRIC EINSTEIN-MAXWELL DILATON-AXION BLACK HOLE

In this section we shall consider the effective Lagrangian of the low-energy heterotic string theory in four dimensions given by [18, 20]

$$\begin{aligned} I &= \frac{1}{16\pi}\int d^4x\sqrt{-g}\Big(R - 2g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi - \frac{1}{2}e^{4\Phi}g^{\mu\nu}\nabla_\mu K_a\nabla_\nu K_a \\ &\quad -e^{-2\Phi}g^{\mu\lambda}g^{\nu\rho}F_{\mu\nu}F_{\lambda\rho} - K_aF_{\mu\nu}\bar{F}^{\mu\nu}\Big), \end{aligned} \tag{1}$$

where the dual of electromagnetic field tensor $F_{\mu\nu}$ is

$$\bar{F}^{\mu\nu} = -\frac{1}{2}\sqrt{-g}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}. \tag{2}$$

Here R is the Riemann curvature scalar, $\varepsilon_{\mu\nu\alpha\beta}$ is the Levi Civita symbol and $g^{\mu\nu}$ is the metric tensor. Also Φ and K_a are the massless dilaton field and the axion field respectively.

In the Boyer-Lindquist coordinates (t, r, θ, φ), the stationary axially-symmetric solution to the Einstein-Maxwell's equations in the presence of the dilaton-axion is given by [18],

$$\begin{aligned} ds^2 &= -\frac{\Sigma - a^2\sin^2\theta}{\Delta}dt^2 - \frac{2a\sin^2\theta}{\Delta}\left[(r^2-2Dr+a^2)-\Sigma\right]dtd\varphi \\ &\quad +\frac{\Delta}{\Sigma}dr^2 + \Delta d\theta^2 + \frac{\sin^2\theta}{\Delta}\left[(r^2-2Dr+a^2)^2 - \Sigma a^2\sin^2\theta\right]d\varphi^2, \end{aligned} \tag{3}$$

where

$$\Delta = r^2 - 2Dr + a^2\cos^2\theta, \qquad \Sigma = r^2 - 2mr + a^2, \tag{4}$$

and

$$e^{2\Phi} = \frac{W}{\Delta} = \frac{\omega}{\Delta}(r^2 + a^2\cos^2\theta), \quad \omega = e^{2\Phi_0}, \tag{5}$$

$$K_a = K_0 + \frac{2aD\cos\theta}{W}, \tag{6}$$

$$A_t = \frac{1}{\Delta}(Qr - ga\cos\theta), \quad A_r = A_\theta = 0, \tag{7}$$

$$A_\varphi = \frac{1}{a\Delta}(-Qra^2\sin^2\theta + g(r^2+a^2)a\cos\theta). \tag{8}$$

The mass M, angular momentum J, electric charge Q, and magnetic charge P, dilaton charge D of the black hole are given by

$$M = m - D, \quad J = a(m-D), \quad Q = \sqrt{2\omega D(D-m)}, \quad P = g. \tag{9}$$

The above results show that the stationary axis symmetric dilaton-axion black hole significantly differs from the the Kerr-Newmann black hole. The two horizons are the inner r_- and the outer one r_+ of the black hole under consideration are

$$r_\pm = M + D \pm \sqrt{(M+D)^2 - a^2}. \tag{10}$$

Only r_+ is the event horizon and one can associate thermodynamical quantities with it.

The Hawking temperature associated with the event horizon is

$$T = \frac{\hbar}{4\pi}\left(\frac{r_+ - M - D}{r_+^2 - 2Dr_+ + a^2}\right). \tag{11}$$

The angular velocity Ω at the event horizon can be rewritten as

$$\Omega = \frac{J/M}{r_+^2 - 2Dr_+ + a^2}. \tag{12}$$

Here J is the angular momentum. The electrostatic potential can be given by

$$\Phi = \frac{-2DM}{Q(r_+^2 - 2Dr_+ + a^2)}. \tag{13}$$

The entropy associated with the event horizon of the dilaton-axion black hole is

$$S = \frac{\pi}{\hbar}(r_+^2 - 2Dr_+ + a^2). \tag{14}$$

III. CARDY-VERLINDE FORMULA AND DILATON-AXION BLACK HOLE

In this section, we introduce the Cardy-Verlinde formula which states that the entropy of a (1+1)-dimensional CFT is given by

$$S = 2\pi\sqrt{\frac{c}{6}\Big(L_0 - \frac{c}{24}\Big)}, \tag{15}$$

where c is the central charge and L_0 is the Virasoro generator. After appropriate identifications of c and L_0, the above Cardy formula, we obtain the generalized Cardy-Verlinde formula which takes the form [1]

$$S_{CFT} = \frac{2\pi R}{\sqrt{a_1 b_1}} \sqrt{E_C(2E - E_C)}, \tag{16}$$

where E is the total energy, E_C is the Casimir energy, a_1 and b_1 are arbitrary positive constants. Also R is the radius of the $n+1$ dimensional spacetime, $ds^2 = -dt^2 + R^2 d\Omega_n$. The definition of Casimir energy is derived by the violation of the Euler relation as

$$E_C = n(E + PV - TS - \Phi Q - J\Omega), \tag{17}$$

where the pressure of the CFT is given by $P = E/nV$. The total energy is the sum of two terms

$$E(S,V) = E_E(S,V) + \frac{1}{2} E_C(S,V). \tag{18}$$

Here E_E is the purely extensive part of the total energy. The Casimir energy and the purely extensive part of the total energy are expressed as

$$E_C = \frac{b_1}{2\pi R} S^{1-\frac{1}{n}}, \tag{19}$$

$$E_E = \frac{a_1}{4\pi R} S^{1+\frac{1}{n}}. \tag{20}$$

IV. ENTROPY OF AXIALLY SYMMETRIC AXION-DILATON BLACK HOLE AND CARDY-VERLINDE FORMULA

Using Eq. (12) with $n = 2$ and $E = M$, we obtain

$$\begin{aligned} E_C &= 3M - 2TS - 2\Phi Q - 2\Omega J, \\ &= 3M - \frac{1}{2}(r_+ - M - D) + \frac{4DM}{r_+^2 - 2Dr_+ + a^2} - \frac{2J^2}{M(r_+^2 - 2Dr_+ + a^2)}. \end{aligned} \tag{21}$$

From (16) we have

$$\begin{aligned} 2E - E_C &= -M + 2TS + 2\Phi Q + 2\Omega J, \\ &= -M + \frac{1}{2}(r_+ - M - D) - \frac{4DM}{r_+^2 - 2Dr_+ + a^2} + \frac{2J^2}{M(r_+^2 - 2Dr_+ + a^2)}. \end{aligned} \tag{22}$$

From (13) and (16), the extensive part of total energy becomes

$$\begin{aligned} E_E &= E - \frac{1}{2}E_C, \\ &= -\frac{1}{2}M + \frac{1}{4}(r_+ - M - D) - \frac{2DM}{r_+^2 - 2Dr_+ + a^2} + \frac{J^2}{M(r_+^2 - 2Dr_+ + a^2)}. \end{aligned} \tag{23}$$

Comparison of (14) and (16) yields

$$\begin{aligned} R &= \frac{b_1 S^{1/2}}{2\pi}\Big[3M - 2TS - 2\Phi Q - 2\Omega J\Big]^{-1}, \\ &= \frac{\frac{b_1}{2\pi}\sqrt{\frac{\pi}{\hbar}(r_+^2 - 2Dr_+ + a^2)}}{3M - \frac{1}{2}(r_+ - M - D) + \frac{4DM}{r_+^2 - 2Dr_+ + a^2} - \frac{2J^2}{M(r_+^2 - 2Dr_+ + a^2)}}. \end{aligned} \tag{24}$$

Comparison of (15) and (18) yields

$$\begin{aligned} R &= \frac{a_1 S^{3/2}}{4\pi}\Big[-\frac{1}{2}M + TS + \Phi Q + \Omega J\Big]^{-1}, \\ &= \frac{\frac{a_1}{4\pi}\Big[\frac{\pi}{\hbar}(r_+^2 - 2Dr_+ + a^2)\Big]^{3/2}}{-\frac{1}{2}M + \frac{1}{4}(r_+ - M - D) - \frac{2DM}{r_+^2 - 2Dr_+ + a^2} + \frac{J^2}{M(r_+^2 - 2Dr_+ + a^2)}}. \end{aligned} \tag{25}$$

Combining the last two expressions (19) and (20), we obtain

$$\begin{aligned} R &= \frac{\sqrt{a_1 b_1}}{2\sqrt{2}}\frac{\pi}{\hbar}(r_+^2 - 2Dr_+ + a^2)\Big[3M - \frac{1}{2}(r_+ - M - D) + \frac{4DM}{r_+^2 - 2Dr_+ + a^2} - \frac{2J^2}{M(r_+^2 - 2Dr_+ + a^2)}\Big]^{-1} \\ &\times\Big[-\frac{1}{2}M + \frac{1}{4}(r_+ - M - D) - \frac{2DM}{r_+^2 - 2Dr_+ + a^2} + \frac{J^2}{M(r_+^2 - 2Dr_+ + a^2)}\Big]^{-1}. \end{aligned} \tag{26}$$

Using (16), (17) and (21) in (11) yields

$$S_{CFT} = \frac{\pi}{\hbar}(r_+^2 - 2Dr_+ + a^2) = S. \tag{27}$$

V. LOGARITHMIC CORRECTION TO THE CARDY-VERLINDE FORMULA

In this section, we shall obtain the first order entropy correction by using corrected Bekenstein-Hawking entropy formula in the Cardy-Verlinde formula. The first order correction to the semi-classical Bekenstein-Hawking entropy S_0 is given by [21]

$$\mathbb{S} = S_0 - \frac{1}{2}\ln C. \tag{28}$$

Here C is the heat capacity of the black hole evaluated at the event horizon. We suppose that $C \simeq S = S_0$ [21] so that the above equation (28) turns out

$$\mathbb{S} = S_0 - \frac{1}{2}\ln S_0. \tag{29}$$

First we calculate the corrected Casimir energy and the corrected extensive part of the total energy by using first order corrected entropy (29) which admit

$$\tilde{E}_C = E_C + T \ln S_0, \tag{30}$$

$$\tilde{E}_E = E - \frac{1}{2}E_C - \frac{1}{2}T \ln S_0. \tag{31}$$

By using modified Casimir energy (30) and the extensive part of the total energy (31) in the Cardy-Verlinde formula (16), we obtain the modified Cardy-Verlinde entropy relation

$$\tilde{S}_0 = \frac{2\pi R}{\sqrt{a_1 b_1}}\sqrt{\tilde{E}_C(2E - \tilde{E}_C)}. \tag{32}$$

Simplifying (32) we obtain

$$\tilde{S}_0 \simeq S_0\Big[1 + \frac{(E - E_C)}{E_C(2E - E_C)}T \ln S_0\Big]. \tag{33}$$

Finally using (33) in (29) yields the corrected entropy as

$$\mathbb{S} \simeq \frac{2\pi R}{\sqrt{a_1 b_1}}\sqrt{E_C(2E - E_C)} + \Big[\frac{2\pi R}{\sqrt{a_1 b_1}}\frac{(E - E_C)}{\sqrt{E_C(2E - E_C)}} - \frac{1}{2}\Big]T \ln \Big[\frac{2\pi R}{\sqrt{a_1 b_1}}\sqrt{E_C(2E - E_C)}\Big]. \tag{34}$$

Hence the entropy correction to the semi-classical Bekenstein-Hawking entropy is obtained in terms of the modified Cardy-Verlinde formula which further investigates the AdS/CFT correspondence in terms of modified Cardy-Verlinde entropy formula. The first term corresponds to the usual CV formula while the second term relates to correction to Hawking entropy in terms of modified Cardy-Verlinde entropy formula.

VI. CONCLUSION

In this paper, we have shown that the Bekenstein-Hawking entropy of the axially-symmetric axion-dilaton black hole can also be expressed in the form of Cardy-Verlinde entropy formula which further investigates the AdS/CFT correspondence in terms of Cardy-Verlinde entropy formula. The axially symmetric dilaton axion black hole is asymptotically flat instead of AdS space. So our study indicates that the AdS/CFT correspondence still holds in the black hole geometries with asymptotically flat background. By using the logarithmic correction to the Bekenstein-Hawking entropy, we obtained the modified expressions for the Casimir and extensive energy relations. By utilizing modified expressions for Casimir

and extensive energy in the Cardy-Verlinde formula, we obtained the corrected S_{CFT} relation which relates the entropy of a certain CFT to its total energy and Casimir energy. The second result of this paper is the entropy correction to the semi-classical Bekenstein-Hawking entropy in terms of the modified Cardy-Verlinde formula. The first term in (34) corresponds to the usual Cardy-Verlinde formula while the second term relates correction to Hawking entropy in terms of modified Cardy-Verlinde entropy formula.

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